

Angular momentum of isolated systems in general relativity^{*}

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Abstract

This article is a conceptual discussion, for non-specialists, of what appears to be a satisfactory solution to the problem of treating angular momentum for isolated radiating systems in general relativity. The approach is a development of one suggested by Penrose, based on twistor theory. While in special relativity angular momentum is a simple tensorial object, in general relativity it acquires components in other representations of the Lorentz group as well. Remarkably, these other components may be identified with the gravitational radiation. Thus special-relativistic angular momentum and gravitational radiation are two parts of one entity, the general-relativistic angular momentum.

Key words: angular momentum, general relativity, gravitational radiation, twistors

1 Introduction

General relativity at once destroys the usual foundations for treating energy, momentum, and angular momentum, and makes the identification of these quantities especially desirable. The gravitational field itself — the varying curvature of space-time — is the obstruction to the existence of isometries, and so to what are usually considered the foundations of the theory of conserved quantities. At the same time, the general coordinate freedom, which is

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introduced to compensate for this, means that it is only invariantly defined quantities which are physically meaningful: energy, momentum and angular momentum would be the most important of these.

The importance of treating conserved quantities in general relativity, and the difficulties in doing so, were recognized by Einstein early on [1]. Einstein was able to overcome these in the restricted case of energy for weak perturbations of Newtonian theory [2]; and the formula he obtained — the famous quadrupole formula for gravitational wave-energy — was deeply influential.

However, the limitations of the quadrupole formula, and other related approximations (“pseudotensor” definitions, short-wave formalisms), are being increasingly felt, particularly with the great amount of analytical and numerical modeling of strongly general-relativistic systems (colliding black holes, etc.), and also with increasingly accurate treatments of more modestly general-relativistic systems (binary pulsars, e.g.) in the past fifteen years or so. One needs to move beyond these approximations.

The question of principle is as important as these practical concerns. What *are* the correct general-relativistic conserved quantities? (In fact, *are* there such quantities? Might we just be asking too much?) We believe that general relativity is a deeper theory than either Newtonian gravity or special relativity, indeed that it subsumes them. If we can find satisfactory general-relativistic definitions of energy, momentum and angular momentum, we may expect these to isolate something quite fundamental.

While the quadrupole formula itself is so closely allied to Newtonian theory as to give us little definitive help with these questions, the issues Einstein took up as he grappled with the problem of conserved quantities remain the fundamental ones today. The issues of what boundary conditions to impose and how to interpret them, the difficulties in formulating a suitable relativistic concept of a closed system, the problem of defining suitable reference frames with respect to which the quantities are defined, remain critical.

It might be tempting to think that issues of boundary conditions and closure are analytic technicalities and that the difficult questions will be answered by “figuring out where to put in the ϵ s,” and that with the right asymptotic conditions the reference frames would be obvious. This would be wrong: progress has required, not only substantial physical insight, but subtle adjustments in what one considers the foundations of the theories to be.

Definitive progress on aspects of these problems occurred only around 1960, with the work of Bondi [3] and Sachs [4]. These authors made precise the concept of an isolated radiating system, and did so by finding a suitable generalization of the Sommerfeld radiation conditions — that is, boundary conditions. They were also able to identify in a subtle way asymptotic reference

frames, and a measure of such a system's energy-momentum, and show that it had very attractive formal properties. Since then, the evidence in favor of the correctness of the Bondi-Sachs definition has accumulated (and there has been no evidence against it); it is now accepted.

(It is accepted, but it hasn't been used that much yet! Despite the considerable activity modeling general-relativistic systems, researchers have not yet succeeded in using the Bondi-Sachs energy-momentum in most cases. The main reason for this is that it is hard to construct the Bondi-Sachs asymptotic regime from the approaches usually used to integrate the equations of motion. This remains an important problem.)

The Bondi-Sachs work gave hope that angular momentum might be accessible by similar techniques. However, this has turned out not to be the case. I will sketch some of this later. But for right now, let me say that the problem is associated with the origin-dependence of angular momentum. It has been very difficult to identify a suitable space of origins on which to base a definition. Using the points in space-time doesn't work, and, as we shall see, the difficulties are bound up very deeply with how gravitational radiation qualitatively affects the structure of space-time.

Another face of the problem is the lack of existence of symmetries — something which (as it happened) could be dodged in the treatment of energy-momentum, but has to be faced squarely for angular momentum. It is the gravitational radiation which destroys symmetry in the asymptotic regime.

As I noted, this lack of symmetries is very serious, because it means that what we usually consider to be the foundations of the theory of angular momentum are absent. What principles will we adopt to guide our search, then? What key properties must some candidate formula have, to be considered angular momentum?

And, supposing we *do* succeed in defining angular momentum general-relativistically, what will the shift in guiding principles mean for our conception of the quantity more generally? If we must alter our views about what angular momentum *is* in order to pass to general relativity, then presumably what we *took* to be angular momentum in Newtonian theory and in special relativity was rather a special case, and what we took to be the foundations were (from the general-relativistic view) rather special properties which hold only in the special case of negligible gravitational effects.

The aim of this article is to provide a conceptual discussion of what appears to be a satisfactory solution to the problem of defining angular momentum for isolated general-relativistic systems. (In fact, the solution has much stronger properties than would have been anticipated on the basis of earlier work.) The technical details of the construction have appeared elsewhere [5].

The main conclusions are:

- (a) There are two, closely related, criteria which should govern the search for angular momentum. The first is that it be possible to formulate a notion of *conservation*. The second is that it should be *universal*, in that there should be a very broad class of systems for which it is definable, and the angular momenta of the different systems in this class should be comparable.

The requirement of conservation might sound like a truism — after all, one refers to angular momentum as a conserved quantity — but in fact the challenge of defining angular momentum has been such that in many proposals for solving it no very useful notion of conservation exists.

The requirement of universality is a very strong restriction which goes to the heart of the difficulties. It is this which rules out using the points in space–time for the origins — there is no preferred way of identifying the points in one space–time with those in another. In fact, the problem of somehow compensating for the loss of a space of origins is the main one to be faced.

- (b) A suitable definition can be given using twistor theory, by developing ideas of Penrose’s [6]. The appearance of twistors is in fact natural, because in twistor theory, the points of space–time appear as secondary, derived, objects, and so the problem of finding origins for angular momentum is recast.
- (c) General-relativistic angular momentum, in the presence of gravitational radiation, has a qualitatively different character than special-relativistic angular momentum. It is represented not simply by a skew two-index tensor field, but contains other representations of the Lorentz group as well. Most remarkably, those “extra” contributions turn out to be precisely a standard measure of the gravitational radiation.

Thus general-relativistic angular momentum unites the tensorial, special-relativistic, angular momentum, with gravitational radiation, which is to be thought of as the essentially general-relativistic portion.

In section 2, I will review the appearance of angular momentum in Newtonian theory and in special relativity. In section 3, more detail on special relativity is given, and in section 4 the treatment of special-relativistic angular momentum by twistors is sketched. Section 5 discusses isolated systems in general relativity, and section 6 how gravitational radiation results in a different asymptotic structure than in special relativity. Section 7 explains the twistorial definition of angular momentum, section 8 explains the sorts of measures of spin and angular momentum which result from this, and section 9 discusses some implications.

This paper is a conceptual, not a technical treatment. For a technical treat-

ment, see ref. [5]. Background reading adequate for understanding the material there is ref. [7].

2 Angular momentum before general relativity

We encounter angular momentum first in Newtonian mechanics, where we learn that for an object of momentum \mathbf{p} at position \mathbf{r} relative to an origin, the (orbital) angular momentum about this origin is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \quad (1)$$

If the object moves in a central force field (central relative, again, to the same origin), this will be conserved. More generally, we learn that the total angular momentum of a closed system will be conserved.

In special relativity, we learn that angular momentum is really a skew two-index tensor field,

$$M_{ab}(x) = \begin{bmatrix} 0 & ER_x & ER_y & ER_z \\ -ER_x & 0 & -L_z & L_y \\ -ER_y & L_z & 0 & -L_x \\ -ER_z & -L_y & L_x & 0 \end{bmatrix} \quad (2)$$

where in addition to the ordinary, spatial, angular momentum the center of energy (R_x, R_y, R_z) (and the energy E itself) appear. This is a very beautiful object, and the center-of-energy (or more properly, the moments ER_x, ER_y, ER_z) appear as the conserved quantities conjugate to the *boosts*, that is, the velocity-changing transformations, just as the spatial angular momentum is conjugate to the rotations. (In equation (2), the x on the left indicates the point in space-time, whereas on the right the subscript x stands for one of the coordinates.)

It's also appropriate to touch on angular momentum in quantum mechanics, for several reasons. In the first place, it is good to remember how strongly angular momentum was bound up with the development of quantum mechanics. It is also true that part of the intuition that led to the Bondi–Sachs construction was actually a product of considerations about quantizing gravity! An important motivation for understanding angular momentum in general relativity remains the hope that this will give us clues about quantum gravity.

In quantum mechanics, we learn that, in the first place, the angular momentum is a wholly new sort of object, an operator. We also learn that there may exist

intrinsic angular momentum, or *spin*, so that the total angular momentum operator is

$$\hat{\mathbf{r}} \times \hat{\mathbf{p}} + \hat{\mathbf{S}}, \quad (3)$$

non-relativistically. (The hats in eq. (3) indicate operators. However, this equation will be the only such case. The symbol $\hat{\mathbf{r}}$ below will indicate a unit vector.) In the remainder of this paper, however, quantum theory, and quantum operators, will play no role.

In fact, while the understanding of spin developed in quantum mechanics, we now realize that the possibility of spin (as intrinsic angular momentum) is subsumed in angular momentum even at a classical level. In modern relativistic parlance, the *spin* is the irreducible, origin-independent, part of the angular momentum (which can be given by the formula

$$S_a = (1/2M)\epsilon_{abcd}M^{bc}P^d \quad (4)$$

with M the mass, P^a the energy-momentum, and ϵ_{abcd} the alternating symbol); we think of M_{ab} as determined jointly by the spin and the center of mass, assuming P_a is known.

3 Special relativity

In order to understand general relativity, we should first review a few elements of special relativity.

Minkowski space $\mathbb{M} = \{(t, x, y, z)\} \cong \mathbb{R}^4$ is the space-time of special relativity; points in it are called *events*, and represent a point in space at an instant in time. The key structure on Minkowski space is its *metric* g_{ab} , in differential form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (5)$$

(in units where the speed of light is $c = 1$). This convention, with one plus and three minuses, is the more physically natural one. The time experienced by an observer traveling a path γ is $\int_{\gamma} ds$. Because essentially all of special relativity can be derived from arguments about comparison of clocks, the metric is central.

A vector v^a in Minkowski space may be classified as *timelike*, *spacelike* or *null*, according to whether $g_{ab}v^a v^b = (v^t)^2 - (v^x)^2 - (v^y)^2 - (v^z)^2$ is positive, negative or zero. Timelike vectors then correspond to speeds less than that of light, spacelike vectors to speeds in excess of light, and (non-zero) null vectors to the speed of light.

Note that timelike (and for that matter, non-zero null) vectors must have

$v^t \neq 0$. Thus the set of these vectors has two components, those which are *future-directed* (with $v^t > 0$), and those which are *past-directed* ($v^t < 0$).

The isometry group of Minkowski space is called the *Poincaré group*.¹ Its structure is formally similar to that of a Euclidean space: it is a semidirect product of the translations and the (relativistic version of the rotations, the) Lorentz group:

$$0 \rightarrow \text{Translations} \rightarrow \text{Poincaré} \rightarrow \text{Lorentz} \rightarrow 0. \quad (6)$$

The fundamental special-relativistic kinematic conserved quantities are the energy-momentum P_a (conjugate to the translations), and the angular momentum $M_{ab}(x)$ (conjugate to the Lorentz motions).

Let's pause to note the important properties of $M_{ab}(x)$. In the first place, it is origin-dependent. In the second-place, its tensorial character means that it is an element of a particular representation of the Lorentz group. The representations we shall be concerned with are labeled by pairs of integers or half-integers (s, j) . In special relativity, the angular momentum takes values in the $s = 1, j = 1$ representation.

(A caution about some confusing terminology: the same representations play important roles in quantum theory, but the interpretation is somewhat different. There, the quantities s and j actually become measures of angular momentum. In this paper, though, they are just labels of representations.)

4 Twistor theory and angular momentum

Twistor theory allows an alternative treatment of special relativity, mathematically equivalent to the usual one, but in which space-time appears as a secondary, derived, concept. (This is the main reason why twistors form a natural candidate for dealing with the difficulties of defining angular momentum in general relativity.)

The twistor space appropriate to special relativity is the space $\mathbb{T} = \{Z^\alpha\} \cong \mathbb{C}^4$ of spinors of the conformal group of Minkowski space; it is naturally equipped with a pseudo-Hermitian norm $Z^\alpha \bar{Z}_\alpha$ of signature $++--$. One can realize \mathbb{T} as a certain space of spinor fields on \mathbb{M} . I should hasten to say that for this article no detailed knowledge of spinors is necessary; it is enough to know that a spinor is a geometric object, something like a vector or a tensor (really

¹ In this paper, reflective motions will play no role. Strictly speaking, we should define the Poincaré and Lorentz groups to consist of orientation- and time-orientation-preserving isometries.

like the square root of a future-directed null vector, and, unlike vectors, by its definition bound to the metric structure of the space-time).

For simplicity, I will only be discussing the geometric interpretation of the *real twistors*, those with $Z^\alpha \bar{Z}_\alpha = 0$, here. Each such twistor corresponds to a null geodesic γ together with a parallel-transported tangent spinor $\pi_{A'}$. I will write $Z^\alpha \leftrightarrow (\gamma, \pi_{A'})$ to indicate this correspondence. Note that the twistor is rather delocalized from the point of view of space-time, being associated with a null geodesic rather than a single point.

The angular momentum of a system is coded in a function $A(Z) = A_{\alpha\beta} Z^\alpha Z^\beta$ on twistor space. In the case of real twistors, the correspondence is very direct:

$$A(Z) = \begin{aligned} &\text{the component of } M_{ab}(x) \text{ determined by the spinor } \pi_{A'} \\ &\text{evaluated at any event } x \text{ on the geodesic } \gamma \end{aligned} \quad (7)$$

if $Z^\alpha \leftrightarrow (\gamma, \pi'_A)$. (The choice of point on γ is immaterial when the component is that corresponding to $\pi_{A'}$.) Explicitly,

$$A(Z) = \begin{aligned} & (8) \\ & \begin{bmatrix} \pi_{0'} & \pi_{1'} \end{bmatrix} \begin{bmatrix} (L_x - ER_x) - i(L_y + ER_x) & -L_z + iER_z \\ -L_z + iER_z & -(L_x + ER_x) + i(-L_y + ER_x) \end{bmatrix} \begin{bmatrix} \pi_{0'} \\ \pi_{1'} \end{bmatrix}, \end{aligned}$$

where $\pi_{0'}$, $\pi_{1'}$ are the components of $\pi_{A'}$ in a standard basis [7], and the center of mass and angular momentum are evaluated at any point x on γ . If we imagine holding the point fixed, but varying the null geodesics and their tangent spinors through it, we can recover from (8) all components of the relativistic angular momentum.

5 General relativity

In general relativity, a *space-time* is a (smooth, connected, paracompact) manifold \mathcal{M} equipped with a Lorentzian metric g_{ab} . At any point, then, we may classify the vectors as timelike, null or spacelike according to the sign of $g_{ab}v^a v^b$. Again, at each point, the timelike and non-zero null vectors divide into two components. We assume that it is possible to choose one such component continuously over the manifold, and that this choice has been made, so that we know which vectors are *future-directed* and which *past-directed*. We also assume \mathcal{M} is oriented.

How can we make precise the idea of an *isolated* system? In some sense, we must say what it means to travel far from the system, and say that in that limit

the system becomes “self-contained.” Roughly speaking, this should mean passing to an appropriate asymptotic regime such that all gravitational effects are localized inside of it.

There are two main approaches to this, based on whether the sense of traveling far from the system means going in spacelike or null directions. (There are also asymptotic regimes in timelike directions, but these do not correspond to a sense of isolation, since in those directions one usually cannot escape from the influences of matter.) The asymptotic spacelike regime might seem, based on non-relativistic experience, most natural, but it does not allow for a direct treatment of radiation, a phenomenon of central interest; it is also less well understood mathematically at present.²

If we wish to study an isolated system and account for the radiation which it emits, we are led to consider moving away from it at the speed of that radiation, here the speed of light, that is, in null directions. This leads to the

Definition. We say a system is *isolated in the sense of Bondi and Sachs* if it is modeled by a space–time for which we may identify a set \mathcal{I}^+ of *escaping wave fronts* with the following properties: (a) each escaping wave front is an equivalence class of asymptotically parallel abreast null geodesics; (b) the set \mathcal{I}^+ has the topology it would for Minkowski space, and can be joined to \mathcal{M} as a hypersurface at infinity; (c) the metric has the same leading asymptotic form (and this holds locally uniformly), as one goes out along these geodesics as it would in Minkowski space.³

Of course, the foregoing is not phrased mathematically precisely, but it does capture the main idea. (The modern technical concept is *weak future asymptotic simplicity* [7].) There are two things worth noting here. First, the definition is in part implicit, because one must identify which wave fronts count as escaping. Second, with this definition we formalize the idea of *modeling* a system by a space–time. That is, the space–time \mathcal{M} is *not* supposed to represent the entire Universe; it is simply a clean mathematical way of representing an idealized system. In practice, we expect many systems to be very well modeled

² A satisfactory definition of angular momentum in this regime was given by Ashtekar and Hansen [8] in the case that the Weyl tensor is “asymptotically electric” there, and Shaw [9] showed that Penrose’s ideas gave this a natural twistorial interpretation. The question of just what the physical significance of the restriction to this case is remains open. It is possible the ideas outlined in this paper could help to lift this restriction, since they do elucidate a parallel issue in the null asymptotic regime.

³ This definition is adapted to treat motion outwards towards the future. One can time-reverse the concepts to treat motion inwards from the past and incoming radiation.

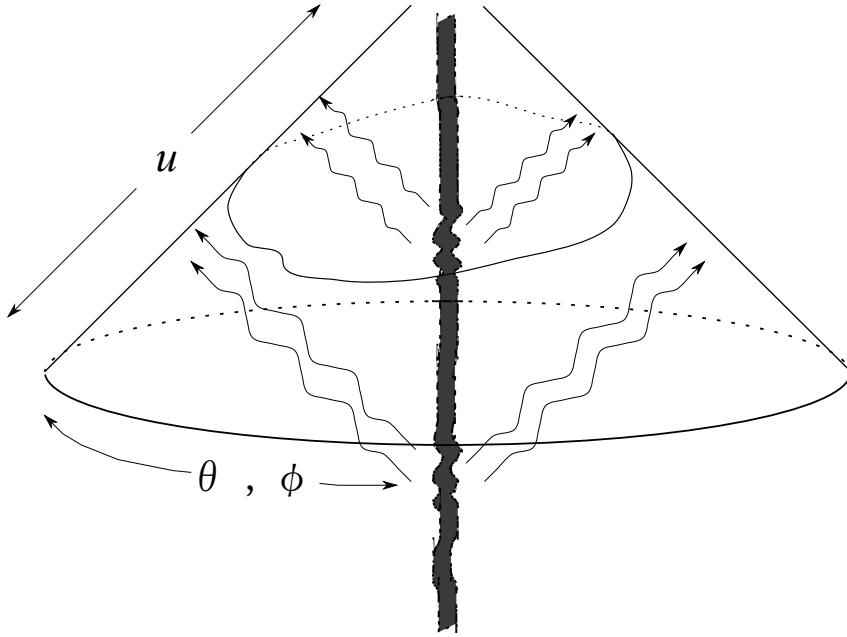


Fig. 1. An isolated system. Future null infinity \mathcal{I}^+ is the half-cone; the space-time is below it. Time increases upwards, space is horizontal (and one spatial dimension is suppressed). The central “world tube” represents a region from which gravitational radiation (indicated by wavy lines) is emitted; these waves leave their profiles on \mathcal{I}^+ . A generic cut $u + \alpha(\theta, \phi) = \text{const}$ is also shown.

in this way. The question of how to account for corrections due to the fact that real systems are not perfectly isolated is, however, a very important and hard one. There is no definitive progress on this (neither for energy, momentum nor angular momentum); this is the problem of *quasi-local* kinematics.

The hypersurface \mathcal{I}^+ at infinity (called *scri-plus*, where “scri” comes by elision from “script I”) has topology $S^2 \times \mathbb{R}$, and this is easily understood. The S^2 labels the asymptotic angles in which the wave-fronts might be directed; the \mathbb{R} labels the times (or more properly, the “retarded times” — what would be $u = t - r$ in Minkowski space) of emission.

The hypersurface \mathcal{I}^+ is evidently the one on which we measure the outgoing wave profiles of any gravitational radiation; at each point of \mathcal{I}^+ (that is, each outgoing wave front), we give a suitable measure of the strength of the wave. It turns out that the strength of the wave is coded in the asymptotic shear of the outgoing null geodesics; this is the *Bondi shear* $\sigma_B(u, \theta, \phi)$.

In radiation problems, we are interested in measuring the energy-momentum and angular momentum in a system after some, but perhaps not all, of the radiation has escaped. This means that we seek measures of the energy-momentum and angular momentum in a system at a retarded time u . However, there is

more to it than that.

6 The Bondi–Metzner–Sachs group

The group of diffeomorphisms preserving the asymptotic structure of an isolated system is the *Bondi–Metzner–Sachs (BMS) group*. Its structure is formally very similar to that of the Poincaré group:

$$0 \rightarrow \text{Supertranslations} \rightarrow \text{BMS} \rightarrow \text{Lorentz} \rightarrow 0 \quad (9)$$

it is a semidirect product of the Lorentz group (which, recall, is the relativistic counterpart of the rotations) and the *supertranslations*. The supertranslations are the motions of the form

$$u \mapsto u + \alpha(\theta, \phi) \quad (10)$$

(and $\theta \mapsto \theta$, $\phi \mapsto \phi$), where α is any smooth function. I will write more about them in a moment.

It is tempting to think that one could use the BMS group as a “stand-in” for the Poincaré group to define angular momentum. Approaches based on this idea have not been considered successful, however. They lead to mathematical objects with uncomfortably large functional degrees of freedom and little structure beyond the purely formal. The underlying reason for this seems to be that, while the Poincaré group is due to the existence of isometries in Minkowski space, the BMS group does not represent isometries, but preservation of a weaker asymptotic structure, which has not been linked to a physically compelling conserved quantity.

The transition from special to general relativity is marked, here, by the expansion of the finite-dimensional group of translations to an infinite-dimensional group of supertranslations. It is on account of this difference that the difficulties in defining angular momentum occur. The most immediately visible effect is that we no longer have a finite-dimensional family of preferred measures of time; for we must consider any retarded time coordinate $u + \alpha(\theta, \phi)$ just as good as u . This means too that we have an infinite-dimensional family of “instants of retarded time,” because any *cut* of \mathcal{I}^+ , of the form $u + \alpha(\theta, \phi) = \text{const}$, must be considered on equal footing with any other (see Fig. 1).

Why are there supertranslations? They come about for a direct physical reason. Imagine a family of observers very far away (“near \mathcal{I}^+ ”) from the system in question, located at various angular coordinates (θ, ϕ) , who synchronize their clocks. Suppose a gravitational wave passes, and then suppose they examine their clocks again. They will in general find that they have become

desynchronized, that is, that they have passed from a common retarded time coordinate u to one of the form $u + \alpha(\theta, \phi)$. It is to accommodate this physical effect that one must introduce supertranslations.

In special relativity, there are no gravitational waves, this sort of desynchronization cannot occur, and consequently one has no need to introduce the BMS group. But for general relativity, where gravitational waves are to be expected, we have no choice.

That all cuts $u + \alpha(\theta, \phi) = \text{const}$ of \mathcal{I}^+ be on equal footing has two sorts of consequences for angular momentum. First, it is bound up with the problem of the absence of a space of origins about which to measure. (In Minkowski space, one can take the $u = \text{const}$ cuts, and their images under translations, as a preferred set of “good” cuts which serve as origins.) Second, it affects the sense in which we quantify how much radiation is left in the space–time after some, but not all, has escaped. Any cut serves as a demarcation of “before” and “after” in this sense. Thus we seek a measure of the angular momentum on any cut: the amount remaining after all radiation prior to that cut has escaped.⁴

I will close this section with a few further comments about the structure of \mathcal{I}^+ . For each fixed (θ, ϕ) , the set of points on \mathcal{I}^+ with differing u -values is called a *generator* of \mathcal{I}^+ . Then the set of generators is the two-sphere S^2 , and \mathcal{I}^+ naturally has the structure of a bundle over S^2 . This is nothing more than saying that there is a well-defined space of asymptotic directions for the asymptotic wave-fronts.

The fact that this space of asymptotic directions is simply S^2 plays a deep role in the analysis. It turns out that the complex structure on S^2 is determined naturally by the asymptotic geometry. Because the Lorentz group acts naturally on this sphere (by fractional linear transformations), one can use structure to define “asymptotically constant” vectors and spinors. This is a beautiful and deep feature of the Bondi–Sachs treatment.

⁴ Thus what will be conserved is the total angular momentum, comprising that remaining in the space–time and that emitted in radiation. One sometimes says that angular momentum at a cut is “conserved, but not absolutely conserved.” While this terminology is a bit odd, it has developed in radiation problems, where indeed absolutely conserved quantities, which are not sensitive to dynamics, are usually not of as much direct interest.

7 Angular momentum in general relativity

The approach that I will discuss to defining angular momentum is a development of ideas of Penrose. Recall that we are interested in defining the angular momentum at a *cut* S of \mathcal{I}^+ (that is, a set of the form $u + \alpha(\theta, \phi) = \text{const}$). Penrose showed that there was a natural definition of a twistor space $\mathbb{T}(S)$ associated with the cut. (The elements of $\mathbb{T}(S)$ are spinor fields on S satisfying certain linear elliptic equations.) The twistor space $\mathbb{T}(S)$ is naturally a four-complex-dimensional vector space, and Penrose found a natural candidate for the angular momentum twistor $A_S(Z)$ on $\mathbb{T}(S)$. (He also suggested a definition of the twistor norm, but pointed out the evidence for this choice was not as strong as one would like.)

This definition had very attractive formal properties. The chief difficulty was that there was no notion of conservation associated with it. The angular momenta at two cuts, S_1 and S_2 , lived in wholly different twistor spaces, $\mathbb{T}(S_1)$ and $\mathbb{T}(S_2)$; there was no way to begin to compare the angular momentum twistors $A_{S_1}(Z)$, $A_{S_2}(Z)$. And physical arguments addressing the supertranslation problem seemed to provide “no-go” theorems, to the effect that there could be no invariant identification of $\mathbb{T}(S_1)$ and $\mathbb{T}(S_2)$ respecting their natural structures (as complex vector spaces equipped with certain other natural twistorial data).

However, it turns out that there *is* a canonical way of identifying the twistor spaces. The trick is that one must be willing to relinquish (most of) their complex and linear structures, and regard them as real manifolds. (The main idea is to show that a spinor field representing a twistor determines in a preferred way a point on the cut, and the twistor can then be specified by data at that point. Those data can then be transported to data at a point on any other cut, by “twistor transport” along the generator of \mathcal{I}^+ through the point. For full details, see [5].)

Since we have a canonical means of identifying the twistor spaces on different cuts, we may say that we have a single twistor space \mathcal{T} . This space is intrinsically a real manifold; and choice S of cut determines a complex-linear structure on \mathcal{T} which identifies it with $\mathbb{T}(S)$. Thus we have one twistor space, with a multiplicity of complex structures.

The space \mathcal{T} also has certain other canonical structures, of which one will be important here. There is a preferred notion of real twistors, and real twistors are identified with real null geodesics meetings \mathcal{I}^+ , equipped with tangent spinors.⁵ As I noted earlier, there *is* a well-defined notion of an asymptotic

⁵ This notion differs from the one derived from the norm suggested by Penrose for $\mathbb{T}(S)$.

spinor.

As to defining angular momentum on \mathcal{T} , we may use Penrose's definition. Thus on each cut S we have an angular momentum $A_S(Z)$. What is different is that we may now regard all these angular momenta, for the different cuts, as functions on the same space \mathcal{T} . We may compute the difference in angular momenta between two cuts simply as $A_{S_1}(Z) - A_{S_2}(Z)$, and we may compute the flux of angular momentum by differentiating with respect to the cut.⁶

There is, however, a novel feature, associated with the change in linear structure as the cut is changed. Penrose's definition gives $A_{S_1}(Z)$ as a quadratic form *with respect to the linear structure* $\mathbb{T}(S_1)$. Since in general $\mathbb{T}(S_2)$ will agree with $\mathbb{T}(S_1)$ as a real manifold but not as a complex vector space, the function $A_{S_1}(Z)$ would appear as a complicated object, not simply a quadratic form, on $\mathbb{T}(S_2)$. If we wish to compare the angular momentum at two different cuts, say $A_{S_1}(Z) - A_{S_2}(Z)$, then we cannot expect this object to appear as a quadratic form on either $\mathbb{T}(S_1)$ or $\mathbb{T}(S_2)$ (or on any other $\mathbb{T}(S)$). It will have a more complicated functional dependence.

This more complicated dependence will appear, when we re-express the twistorial formulas in more conventional terms, as the angular momentum not simply taking values in the $s = 1, j = 1$ representation of the Lorentz group, but having components also in the $s = 1, j \geq 2$ (for integral j) representations.

8 Spin and center of mass

I mentioned earlier that in special relativity the angular momentum comprises two parts, the spin and the center of mass. It is quite remarkable that the general-relativistic angular momentum can be decomposed in the same way, with attractive results.

The general-relativistic spin is not simply a vector, but a quantity which varies over the sphere. It can be given as

$$\text{Spin}(\hat{\mathbf{r}}) = \mathbf{s}_v \cdot \hat{\mathbf{r}} + M \Im \lambda(\hat{\mathbf{r}}), \quad (11)$$

where $\hat{\mathbf{r}}$ is a unit vector representing the direction the spin is to be measured

⁶ Thus angular momentum is conserved in the sense that we may now define the angular momentum emitted in gravitational radiation between two cuts to be the difference in angular momenta on the cuts. In this sense, the conservation is rather trivial. What is not at all trivial, however, is the *basis* for that statement, the fact that we have a well-defined way of comparing the angular momenta on arbitrary cuts.

in, the vectorial part of the spin is \mathbf{s}_v , the mass is M , and $\Im\lambda$ is a measure of the “magnetic” part of the Bondi shear.⁷ That is, the spin can be measured in any direction $\hat{\mathbf{r}}$, but these “components” $\text{Spin}(\hat{\mathbf{r}})$ do not “integrate up” to give simply a vector, but rather a more complicated quantity. Investigations of angular momentum in general relativity have repeatedly encountered this magnetic part of the shear, but its role has been difficult to pin down. Here we see it is simply the general-relativistic part of the specific spin. (“Specific” meaning per unit mass.)

While formulas exist for the center of mass, its properties can be explained more satisfactorily in geometric language. As the vector $\hat{\mathbf{r}}$ varies over the sphere of directions on the cut in question, the definition gives, for each $\hat{\mathbf{r}}$, a null vector inwards from the cut which is to be thought of as directed towards the center of mass of the system. The non-linear part of the dependence of this vector on $\hat{\mathbf{r}}$ turns out to be given precisely by $\Re\lambda(\hat{\mathbf{r}})$, a measure of the “electric” part of the Bondi shear.

The picture is actually a bit stronger. For a system which is asymptotically stationary, one can use the supertranslation freedom to fix a Bondi retarded time parameter u for which the electric part of the shear is zero. The description of center of mass in the present approach makes the cut at which we measure look like a “snapshot” of such a stationary system, taking into account that the cut may not be a $u = \text{const}$ one for the retarded time parameter associated with the stationarity. This is an appealing physical picture, which seems to be as strong as one could expect.

What is most remarkable is that, for both the spin and the center of mass, the essentially general-relativistic portion of the angular momentum is the Bondi shear, which is a measure of the gravitational radiation. We may thus say that general-relativistic angular momentum embraces two parts, a special-relativistic ($j = 1$) contribution, and the gravitational radiation ($j \geq 2$).

9 Implications

We have seen that, in passing from special to general relativity, it is natural to identify as the angular momentum an object which at once subsumes the special-relativistic ($j = 1$) angular momentum and the gravitational radiation ($j \geq 2$). As one might expect, in situations where general relativity is weak, one gets to good approximation the conventional, special-relativistic angular momentum and one can ignore the general-relativistic terms. But in general

⁷ Just as light waves have electric and magnetic components, so gravitational waves have what are called gravitoelectric and gravitomagnetic components.

one must include the corrections.

It might be helpful, conceptually, to consider what happens if several billiard balls collide. For all practical purposes this is a Newtonian problem, and one can use non-relativistic angular momentum to analyze it to excellent accuracy. However, in principle, when the balls collide they give off small bursts of gravitational radiation. These give rise to small supertranslations in the measurements of time before and after the collisions, and thus to small ambiguities in the comparison of the Newtonian (or special-relativistic) angular momenta before and after the collisions. The general-relativistic definition, however, resolves these ambiguities.

For billiard balls, the ambiguities are tiny beyond measurement ($\sim 10^{-53}$ s), but of course they were just to give a homely example. In (say) the close scattering of two black holes, the resolution of the ambiguities would be a far more serious matter, necessary to give any quantitative meaning to angular momentum.

One interesting consequence of this analysis is that it turns out that angular momentum may be emitted at *first order* in the gravitational wave strength, whereas ordinary energy–momentum is only radiated at second order. (Even the $j = 1$ part of the angular momentum can be radiated at first order, although one needs highly asymmetric and rapidly-changing systems for this to occur.) This means that the emission, absorption, or exchange of angular momentum via gravitational waves may be a much more important feature of ordinary (not strongly radiating) general relativistic systems, than the corresponding phenomena for energy–momentum.

While the treatment of angular momentum suggested here does appear to be satisfactory, in the sense that it is natural and has attractive features, it leads to deeper questions, which at present we do not have answers for: Why should the angular momentum have this form? What underlying structure — substituting for the isometries in special relativity — is responsible for the existence of angular momentum in general relativity?

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